

# Conformal Vector Fields of a Class of Finsler Spaces II

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## Abstract

In this paper, we first give two fundamental principles under a technique to characterize conformal vector fields of  $(\alpha, \beta)$  spaces to be homothetic and determine the local structure of those homothetic fields. Then we use the principles to study conformal vector fields of some classes of  $(\alpha, \beta)$  spaces under certain curvature conditions. Besides, we construct a family of non-homothetic conformal vector fields on a family of locally projectively Randers spaces.

**Keywords:** Conformal (Homothetic) vector field,  $(\alpha, \beta)$ -space, S-curvature, Douglas metric, Projective flatness

**MR(2000) subject classification:** 53B40

## 1 Introduction

Let  $F$  be a Finsler metric on a manifold  $M$  with the fundamental metric matrix  $(g_{ij})$ , and  $V$  be a vector field on  $M$ .  $V$  is called a conformal vector field of the Finsler manifold  $(M, F)$  if  $V = V^i \partial / \partial x^i$  satisfies

$$V_{0|0} = -2cF^2, \quad (1)$$

where the symbol  $|$  denotes the horizontal covariant derivative with respect to Cartan, or Berwald or Chern connection, and  $V_{0|0} = V_{i|j} y^i y^j$ ,  $V_i = g_{im} V^m$  and  $c$ , called a conformal factor, is a scalar function on  $M$ . If  $c = \text{constant}$ , then  $V$  is called homothetic. If  $c = 0$ ,  $V$  is called a Killing vector field. An equivalent description of (1) is (11) below ([6]).

Conformal vector fields (esp. Killing vector fields) play an important role in Finsler geometry. When  $F$  is a Riemann metric, it is shown that the local solutions of a conformal vector field can be determined on  $(M, F)$  if  $(M, F)$  is of constant sectional curvature in dimension  $n \geq 3$  ([10] [12]), or more generally locally conformally flat in dimension  $n \geq 2$  ([16]), or under other curvature conditions ([17]). Some important problems in Finsler geometry can be solved by constructing a conformal vector field of a Riemann metric with certain curvature features (especially of constant sectional curvature) (cf. [2] [18]–[20] [22]).

A vector field on a manifold  $M$  induces a flow  $\varphi_t$  acting on  $M$ , and  $\varphi_t$  is naturally lifted to a flow  $\tilde{\varphi}_t$  on the tangent bundle  $TM$ , where  $\tilde{\varphi}_t : TM \mapsto TM$  is defined by  $\tilde{\varphi}_t(x, y) = (\varphi_t(x), \varphi_{t*}(y))$ , where  $x \in M, y \in T_x M$ . In [9], Huang-Mo define a homothetic vector field on a Finsler space by

$$\tilde{\varphi}_t^* F = e^{-2ct} F, \quad (2)$$

where  $c$  is a constant. Then in [9], it obtains the relation between the flag curvatures of two Finsler metric  $F$  and  $\tilde{F}$ , where  $\tilde{F}$  is defined by  $(F, V)$  under navigation technique for a homothetic vector field  $V$  of  $F$ . Note that for a scalar function  $c$ , (1) does not imply (2). We show in [21] that for a vector field  $V$  and a scalar function  $c$  on a Finsler manifold  $(M, F)$ , if (2) holds, then  $c(\varphi_t(x)) = c(x)$ ; and (1) implies (2) iff.  $c(\varphi_t(x)) = c(x)$ .

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\*Supported by the National Natural Science Foundation of China (11471226)

An  $(\alpha, \beta)$ -metric is defined by a Riemannian metric  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and a 1-form  $\beta = b_i(x)y^i$  on a manifold  $M$ , which can be expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where  $\phi(s)$  is a function satisfying certain conditions such that  $F$  is regular. In [13], Shen-Xia study conformal vector fields of Randers spaces under certain curvature conditions. In [7], Kang characterizes conformal vector fields of  $(\alpha, \beta)$ -spaces by some PDEs in a special case  $\phi'(0) \neq 0$ . Later on, we prove the same result for all non-Riemannian  $(\alpha, \beta)$ -spaces (without the condition  $\phi'(0) \neq 0$ ) ([21]). In this paper, we mainly show some curvature conditions of some  $(\alpha, \beta)$ -spaces on which every conformal vector field must be homothetic, and also consider the possible local solutions for those homothetic vector fields.

First, we show two fundamental results on conformal vector fields to be homothetic and the local solutions for homothetic vector fields respectively.

**Theorem 1.1** *Let  $F = \alpha\phi(\beta/\alpha)$  be an  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$ , where  $\phi(s)$  satisfies  $\phi(s) \neq \sqrt{1+ks^2}$  for any constant  $k$  and  $\phi(0) = 1$ . Define a Riemannian metric  $h = \sqrt{h_{ij}(x)y^i y^j}$  and a 1-form  $\rho = p_i(x)y^i$  by*

$$h = \sqrt{u(b^2)\alpha^2 + v(b^2)\beta^2}, \quad \rho = w(b^2)\beta, \quad (b^2 := \|\beta\|_\alpha^2), \quad (3)$$

where  $u = u(t) \neq 0, v = v(t), w = w(t) \neq 0$  are some smooth functions. Suppose  $\rho$  is a conformal 1-form of  $h$ . Then any conformal vector field of  $(M, F)$  must be homothetic.

**Theorem 1.2** *In Theorem 1.1, further suppose the dimension  $n \geq 3$  and  $h$  is of constant sectional curvature  $\mu$ . Locally we can express*

$$h = \frac{2}{1 + \mu|x|^2}|y|, \quad \rho = \frac{4}{(1 + \mu|x|^2)^2} \left\{ -2(\lambda + \langle d, x \rangle) \langle x, y \rangle + |x|^2 \langle d, y \rangle + p_r^i x^r y^i + \langle e, y \rangle \right\}, \quad (4)$$

where  $\lambda$  is a constant number,  $d, e$  are constant vectors and  $P = (p_i^j)$  is a skew-symmetric matrix. Let  $V = V^i(x)\partial/\partial x^i$  be a conformal vector field of  $(M, F)$  with the conformal factor  $c$ . Then we have one of the following cases:

(i)  $(\mu = 0, \lambda = 0, d = 0)$   $V$  is given by

$$V^i = -2\tau x^i + q_m^i x^m + \gamma^i, \quad (5)$$

$$(Qe = P\gamma, PQ - QP = 2\tau P). \quad (6)$$

In this case, we have  $c = \tau$ .

(ii)  $(\mu = 0, \lambda \neq 0)$   $V$  is given by

$$V^i = q_m^i x^m + \gamma^i, \quad (7)$$

$$(\langle d, \gamma \rangle = 0, Qd = 0, Qe = -2\lambda\gamma + P\gamma, PQ - QP = 2R), \quad (8)$$

where  $R = (r_j^i)$  is defined by  $r_j^i = \gamma^i d^j - \gamma^j d^i$ . In this case, we have  $c = 0$ , and so  $V$  is a Killing vector field.

(iii)  $(\mu \neq 0)$   $V$  is given by

$$V^i = 2\mu\langle \gamma, x \rangle x^i + (1 - \mu|x|^2)\gamma^i + q_m^i x^m, \quad (9)$$

$$(\langle d + \mu e, \gamma \rangle = 0, Qd = -\mu(2\lambda\gamma + P\gamma), Qe = -2\lambda\gamma + P\gamma, PQ - QP = 2R), \quad (10)$$

where  $R = (r_j^i)$  is defined by  $r_j^i = \mu(e^i \gamma^j - e^j \gamma^i) + \gamma^i d^j - \gamma^j d^i$ . In this case, we have  $c = 0$ , and so  $V$  is a Killing vector field.

In (5)–(10),  $\tau$  is a constant number,  $\gamma = (\gamma^i)$  is a constant vector and  $Q = (q_k^i)$  is a constant skew-symmetric matrix.

Note that for  $\lambda \neq 0$  in (10), we have

$$Qd = -\mu(2\lambda\gamma + P\gamma), \quad Qe = -2\lambda\gamma + P\gamma \implies \langle d + \mu e, \gamma \rangle = 0.$$

In Theorem 1.2, if additionally  $\rho$  is closed, then  $V$  is given by (5)–(10) with  $P = 0$  and  $d = \mu e$ . This case has actually been obtained in [21]. See Corollary 4.1 below.

In Theorem 1.2, if  $\rho$  is a homothetic 1-form of  $h$ , then  $V$  is given by (5)–(10) with  $d = 0$  and  $\mu = 0$ , or  $d = -\mu e$  and  $\lambda = 0$ . See Corollary 4.2 below.

In Theorem 1.2, if the condition that  $\rho$  is a conformal 1-form of  $h$  is replaced by the condition that  $\rho$  is closed, then a conformal vector field of  $(M, F)$  is not necessarily homothetic (see Remark 7.3). This implies that for a locally projectively flat Randers space  $(M, F)$  with  $F = \alpha + \beta$ , a conformal vector field of  $(M, F)$  is not necessarily homothetic (cf. [21]). We will show a detailed construction of such a family of examples in Section 7.

Under some certain curvature conditions of an  $(\alpha, \beta)$ -space, usually we can define  $h$  and  $\rho$  by (3) by choosing suitable functions  $u, v$  and  $w$ , such that  $\rho$  is a conformal 1-form of  $h$ , or  $\rho$  is a conformal 1-form of  $h$  and  $h$  is of constant sectional curvature. Following this idea, in Sections 5–7, as an application of Theorem 1.1 and Theorem 1.2, we will study some properties of conformal vector fields of some special  $(\alpha, \beta)$  spaces under certain curvature conditions. In Section 5, we study conformal vector fields on  $(\alpha, \beta)$ -spaces of isotropic S-curvature, and our main result is Theorem 5.1. In Section 6, we study conformal vector fields on  $(\alpha, \beta)$ -spaces of Douglas type, and our main result is Theorem 6.1. In Section 7, we study conformal vector fields on locally projectively flat Randers spaces, and our main result is Example 7.2.

## 2 Preliminaries

Let  $F$  be a Finsler metric on a manifold  $M$ , and  $V$  be a vector field on  $M$ . Let  $\varphi_t$  be the flow generated by  $V$ . Define  $\tilde{\varphi}_t : TM \mapsto TM$  by  $\tilde{\varphi}_t(x, y) = (\varphi_t(x), \varphi_{t*}(y))$ . For a conformal vector field  $V$  defined by (1), Huang-Mo show in [6] an equivalent definition in the way

$$\tilde{\varphi}_t^* F = e^{-2\sigma_t} F, \quad (11)$$

where  $\sigma_t$  is a function on  $M$  for every  $t$ , and in this case,  $c$  in (1) and  $\sigma_t$  in (11) are related by

$$\sigma_t = \int_0^t c(\varphi_s) ds, \quad c = \frac{d}{dt} \Big|_{t=0} \sigma_t.$$

Now in a special case of (11), suppose  $\tilde{\varphi}_t^* F = e^{-2ct} F$ , namely,

$$F(\varphi_t(x), \varphi_{t*}(y)) = e^{-2c(x)t} F(x, y), \quad (12)$$

where  $c$  is a scalar function on  $M$ . Differentiating (12) by  $t$  at  $t = 0$ , we obtain

$$X_V(F) = -2cF, \quad (13)$$

where

$$X_V := V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j} \quad (14)$$

is a vector field on  $TM$ . However, (13) generally does not imply (12) for a scalar function (see [21]). We have a different description for conformal vector fields.

**Lemma 2.1** ([21]) *A vector field  $V$  on a Finsler manifold  $(M, F)$  is conformal iff.*

$$X_V(F^2) = -4cF^2 \iff X_V(F) = -2cF, \quad (15)$$

where  $c$  is a scalar function on  $M$ .

For conformal vector fields on a Riemann manifold, we have the following lemma.

**Lemma 2.2** ([16] [21]) *Let  $\alpha$  be a Riemann metric of constant sectional curvature  $\mu$  on an  $n$ -dimensional manifold  $M$ . Locally express  $\alpha$  by*

$$\alpha = \frac{2}{1 + \mu|x|^2}|y|.$$

- (i) ( $n \geq 3$ ) *Let  $V$  be a conformal vector field of  $(M, \alpha)$  with the conformal factor  $c = c(x)$ . Then locally we have*

$$V^i = -2(\lambda + \langle d, x \rangle)x^i + |x|^2 d^i + q_r^i x^r + \eta^i, \quad c = \frac{\lambda(1 - \mu|x|^2) + \langle \mu\eta + d, x \rangle}{1 + \mu|x|^2},$$

where  $\lambda$  is a constant number,  $d, \eta$  are constant vectors and  $(q_i^j)$  is skew-symmetric.

- (ii) ( $n \geq 2$ ) *In (i), if additionally the 1-form  $V_i y^i$  is closed ( $V_i := a_{ik} V^k$ ), then locally we have*

$$V^i = -2(\lambda + \mu\langle e, x \rangle)x^i + (1 + \mu|x|^2)e^i, \quad c = \frac{\lambda(1 - \mu|x|^2) + 2\mu\langle e, x \rangle}{1 + \mu|x|^2}.$$

For an  $(\alpha, \beta)$  space  $(M, F)$  with  $F = \alpha\phi(\beta/\alpha)$ , a conformal vector field is characterized by the following lemma, which is also proved in [7] by assuming  $\phi'(0) \neq 0$ .

**Lemma 2.3** ([21]) *Let  $F = \alpha\phi(\beta/\alpha)$  be an  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$ , where  $\phi(s)$  satisfies  $\phi(s) \neq \sqrt{1 + ks^2}$  for any constant  $k$  and  $\phi(0) = 1$ . Then  $V = V^i(x)\partial/\partial x^i$  is a conformal vector field of  $(M, F)$  with the conformal factor  $c = c(x)$  if and only if*

$$V_{i;j} + V_{j;i} = -4ca_{ij}, \quad V^j b_{i;j} + b^j V_{j;i} = -2cb_i, \quad (16)$$

where  $V_i$  and  $b^i$  are defined by  $V_i := a_{ij}V^j$  and  $b^i := a^{ij}b_j$ , and the covariant derivatives are taken with respect to the Levi-Civita connection of  $\alpha$ .

In this paper, for a Riemannian metric  $\alpha = \sqrt{a_{ij}y^i y^j}$  and a 1-form  $\beta = b_i y^i$ , let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij},$$

where we define  $b^i := a^{ij}b_j$ ,  $(a^{ij})$  is the inverse of  $(a_{ij})$ , and  $\nabla\beta = b_{i|j}y^i dx^j$  denotes the covariant derivatives of  $\beta$  with respect to  $\alpha$ .

### 3 Proof of Theorem 1.1

To prove Theorem 1.1, we first show two lemmas in Riemann geometry.

**Lemma 3.1** *The Lie bracket of two conformal vector fields of a Riemann manifold is a conformal vector.*

**Lemma 3.2** *Let  $V$  and  $cV$  be two conformal vector fields of a Riemann manifold, where  $c = c(x)$  is a scalar function on the manifold. Then  $c$  must be a constant.*

In [6], Huang-Mo have generalized Lemmas 3.1 and 3.2 to general Finsler manifolds.

For a pair  $(\alpha, \beta)$ , define a new pair  $(h, \rho)$  by (3), we obtain in [21], an equivalent characterization of (16) in terms of  $(h, \rho)$  as follows.

**Lemma 3.3** [21] *Let  $\alpha = \sqrt{a_{ij}y^iy^j}$  be a Riemann metric and  $\beta = b_iy^i$  be a 1-form and  $V = V^i\partial/\partial x^i$  be a vector field on an  $n$ -dimensional manifold  $M$ . Define a Riemann metric  $h = \sqrt{h_{ij}(x)y^iy^j}$  and a 1-form  $\rho = p_i(x)y^i$  by (3), namely,*

$$h = \sqrt{u(b^2)\alpha^2 + v(b^2)\beta^2}, \quad \rho = w(b^2)\beta, \quad (b^2 := \|\beta\|_\alpha^2),$$

where  $u = u(t) \neq 0, v = v(t), w = w(t) \neq 0$  are some smooth functions. Then  $\alpha, \beta$  and  $V$  satisfy (16) if and only if

$$V_{0|0} = -2ch^2, \quad V^j p_{i|j} + p^j V_{j|i} = -2cp_i, \quad (17)$$

where  $p^j := h^{ij}p_i$ ,  $V_j := h_{ij}V^i$  and the covariant derivative is taken with respect to the Levi-Civita connection of  $h$ .

Now we show the proof of Theorem 1.1. Since  $V$  is a conformal vector field of the  $(\alpha, \beta)$  space  $(M, F)$  with  $F = \alpha\phi(\beta/\alpha)$ , by Lemma 2.3,  $\alpha, \beta$  and  $V$  satisfy (16) with the conformal factor  $c = c(x)$ . Then by Lemma 3.3,  $h, \rho$  and  $V$  satisfy (17). The first formula in (17) is equivalent to  $V_{i|j} + V_{j|i} = -4ch_{ij}$ , and then using this, a simple observation shows that the second formula in (17) is equivalent to

$$[V, W] = 2cW,$$

where  $W = W^i\partial/\partial x^i$  is a vector field defined by  $W^i := h^{ij}p_j$ . The first formula in (17) shows that  $V$  is a conformal vector field of  $h$ , and by assumption,  $W$  is also a conformal vector field of  $h$ . Now by Lemma 3.1,  $[V, W]$  is a conformal vector field of  $h$ , and so is  $cW$ . Thus by Lemma 3.2, we have  $c = \text{constant}$ , which means that  $V$  is a homothetic vector field of  $(M, F)$ . Q.E.D.

## 4 Proof of Theorem 1.2

In Theorem 1.2, since  $\rho$  is a conformal 1-form of  $h$  and  $V$  is a conformal vector field of  $(M, F)$  with the conformal factor  $c$ , we have  $c = \text{constant}$  by Theorem 1.1. Actually, we can also directly verify  $c = \text{constant}$  in giving the local solution of  $V$  under the assumption of Theorem 1.2.

Since  $\rho$  is a conformal 1-form of  $h$  and  $V$  is a conformal vector field of  $(M, F)$ , we have (17) by Lemma 2.3 and Lemma 3.3. By the assumption that the dimension  $n \geq 3$  and  $h$  is of constant sectional curvature  $\mu$ , it follows from Lemma 2.2 (i) that locally we can write

$$p_i = \frac{4}{(1 + \mu|x|^2)^2} \left\{ -2(\lambda + \langle d, x \rangle)x^i + |x|^2 d^i + p_r^i x^r + e^i \right\}, \quad (18)$$

and correspondingly  $V$  and  $c$  are expressed as

$$V^i = -2(\tau + \langle \eta, x \rangle)x^i + |x|^2 \eta^i + q_r^i x^r + \gamma^i, \quad (19)$$

$$c = \frac{\tau(1 - \mu|x|^2) + \langle \mu\gamma + \eta, x \rangle}{1 + \mu|x|^2}, \quad (20)$$

where  $\lambda, \tau$  are a constant numbers,  $d, e, \eta, \gamma$  are constant vectors and  $P = (p_j^i), Q = (q_i^j)$  are skew-symmetric constant matrices. The second equation in (17) is equivalent to

$$V^j \frac{\partial p_i}{\partial x^j} + p_j \frac{\partial V^j}{\partial x^i} = -2cp_i. \quad (21)$$

Now plugging (18), (19) and (20) into (21) yields an equivalent equation of a polynomial in  $(x^i)$  of degree four, in which every degree must be zero. Then we respectively have (from degree zero to degree four)

$$Qe = -2\lambda\gamma + P\gamma, \quad (22)$$

$$0 = 2(\langle \eta, e \rangle - \langle d, \gamma \rangle + 2\lambda\tau)x^i - 2(\langle e, x \rangle \eta^i + \langle d, x \rangle \gamma^i + \mu \langle \gamma, x \rangle e^i - \langle \gamma, x \rangle d^i) + (p_j^i q_k^j - q_j^i p_k^j - 2\tau p_k^i)x^k, \quad (23)$$

$$0 = [2\lambda(\eta^i - \mu\gamma^i) - \mu q_k^i e^k + 4\tau(\mu e^i - d^i) - q_k^i d^k + p_k^i \eta^k + \mu p_k^i \gamma^k]|x|^2 + 2[2\lambda\mu \langle \gamma, x \rangle + 4\tau \langle d, x \rangle - \langle d, Qx \rangle + \langle \eta, Px \rangle]x^i - 2\langle \mu\gamma + \eta, x \rangle p_k^i x^k, \quad (24)$$

$$0 = \left\{ 2\mu(\langle \eta, e \rangle - \langle d, \gamma \rangle - 2\lambda\tau)x^i - 2\mu \langle e, x \rangle \eta^i - 2\mu \langle d, x \rangle \gamma^i + 2\langle \eta, x \rangle (\mu e^i - d^i) + \mu(p_j^i q_k^j - q_j^i p_k^j + 2\tau p_k^i)x^k \right\} |x|^2 + 4\langle d, x \rangle \langle \eta + \mu\gamma, x \rangle x^i, \quad (25)$$

$$0 = \mu[(2\lambda\eta^i - q_k^i d^k + p_k^i \eta^k)|x|^2 - 2\langle 2\lambda\eta - Qd + P\eta, x \rangle x^i]. \quad (26)$$

We will determine  $V$  from (22)–(26). First we will show that  $\mu = 0$  and  $\eta = 0$ , or  $\tau = 0$  and  $\eta = -\mu\gamma$ . Then by (20) we have  $c = \text{constant}$  given by  $c = \tau$  or  $c = 0$ .

**Case I :** Assume  $\mu = 0$ . We will prove that  $\eta = 0$  from (22)–(26). Let  $\eta \neq 0$  in the following discussion for this case. We show a contradiction.

Plugging  $\mu = 0$  into (25) yields

$$-2\langle \eta, x \rangle (d^i |x|^2 - 2\langle d, x \rangle x^i) = 0,$$

which implies  $d = 0$ . Plugging  $\mu = 0$  and  $d = 0$  into (24) yields

$$(p_k^i \eta^k + 2\lambda\eta^i)|x|^2 - 2(\langle P\eta, x \rangle x^i + \langle \eta, x \rangle p_k^i x^k) = 0. \quad (27)$$

Contracting (27) by  $x^i$  yields

$$|x|^2 (\langle 2\lambda\eta - P\eta, x \rangle) = 0,$$

which implies  $P\eta = 2\lambda\eta$ . Since  $P$  is real and skew-symmetric, its real characteristic roots must be zeros. Thus it follows from  $P\eta = 2\lambda\eta$  that  $\lambda = 0$  and  $P\eta = 0$ . Now plugging  $\lambda = 0$  and  $P\eta = 0$  back into (27) gives  $\langle \eta, x \rangle p_k^i x^k = 0$ . So we get  $P = 0$ . Further, plugging  $\mu = 0, d = 0, \lambda = 0$  and  $P = 0$  into (23), and then contracting it by  $x^i$  we get

$$\langle \eta, e \rangle |x|^2 - \langle \eta, x \rangle \langle e, x \rangle = 0,$$

which shows that  $e = 0$ . Now since  $d = 0, \lambda = 0, e = 0$  and  $P = 0$ , we obtain  $\beta = \rho = 0$  from (18). Thus  $F = \alpha\phi(\beta/\alpha) = \alpha$  is Riemannian. It is a contradiction.

**Case II :** Assume  $\mu \neq 0$ . We will prove that  $\eta = -\mu\gamma$  and  $\tau = 0$  from (22)–(26).

**Case IIa :** Suppose  $\eta \neq -\mu\gamma$ . We will show a contradiction.

By (26) we get

$$Qd = 2\lambda\eta + P\eta. \quad (28)$$

Contracting (23) by  $x^i$  we get

$$(2\lambda\tau - \langle d, \gamma \rangle + \langle \eta, e \rangle)|x|^2 - \langle e, x \rangle \langle \eta + \mu\gamma, x \rangle = 0,$$

which implies  $e = 0$  (since  $\eta \neq -\mu\gamma$ ). By (25), it directly shows that  $d = 0$  (since  $\eta \neq -\mu\gamma$ ). Now by (28), we have  $P\eta = -2\lambda\eta$  which implies

$$\eta = 0, \text{ or } \lambda = 0 \text{ and } P\eta = 0, \quad (29)$$

since  $P$  is skew-symmetric. Similarly, by (22), we have  $P\gamma = 2\lambda\gamma$  which implies

$$\gamma = 0, \text{ or } \lambda = 0 \text{ and } P\gamma = 0. \quad (30)$$

Since  $\eta \neq -\mu\gamma$ , by (29) and (30) we get  $\lambda = 0, P\eta = 0$  and  $P\gamma = 0$ . By this fact and  $d = 0, e = 0$ , we obtain  $-2\langle \eta + \mu\gamma, x \rangle Px = 0$  from (24), which shows  $P = 0$ . Then again we get a contradiction since  $F$  is Riemannian in this case.

**Case IIb :** By the discussion in Case IIa, we have obtained  $\eta = -\mu\gamma$ . Suppose  $\tau \neq 0$ . We will get a contradiction.

Plug  $\eta = -\mu\gamma$  into (24), we get

$$(4\mu\tau e^i - \mu q_k^i e^k - 4\mu\lambda\gamma^i - 4\tau d^i - q_k^i d^k)|x|^2 + 2\langle 4\tau d + 2\mu\lambda\gamma + Qd + \mu P\gamma, x \rangle x^i = 0,$$

from which we obtain

$$Qd = 4\mu\tau e - \mu Qe - 4\mu\lambda\gamma - 4\tau d, \quad (31)$$

$$Qd = -4\tau d - 2\mu\lambda\gamma - \mu P\gamma. \quad (32)$$

Now plugging (22) into (31)–(32) gives  $4\mu\tau e = 0$ , which shows  $e = 0$ . Then by (22) we have  $P\gamma = 2\lambda\gamma$ , which implies  $\gamma = 0$ , or  $\lambda = 0$  and  $P\gamma = 0$ . In either case, we have  $Qd = -4\tau d$  from (32). Thus we obtain  $d = 0$  since  $\tau \neq 0$ . Plugging  $d = e = 0$  into (23) and contracting it by  $x^i$  we get  $4\lambda\tau|x|^2 = 0$ , which shows  $\lambda = 0$ . Now we have proved

$$\lambda = 0, \quad d = 0, \quad e = 0. \quad (33)$$

Plug (33) into (23) and (25), we respectively obtain

$$PQx - QPx - 2\tau Px = 0, \quad (34)$$

$$\mu(PQx - QPx + 2\tau Px)|x|^2 = 0. \quad (35)$$

Then (35)/ $(\mu|x|^2) - (34)$  yields  $4\tau Px = 0$ , from which we get  $P = 0$ . So  $F$  is Riemannian again by (33) and  $P = 0$ . This is a contradiction.

Therefore, we have  $\mu = 0$  and  $\eta = 0$ , or  $\eta = -\mu\gamma$  and  $\tau = 0$  by the discussion in Case I and Case II above. Then we will solve  $V$  from (22)–(26) in the following.

**Case A :** Assume  $\mu = 0$  and  $\eta = 0$ . Plugging  $\mu = 0$  and  $\eta = 0$  into (23) gives

$$2(2\lambda\tau - \langle d, \gamma \rangle)x^i - 2\tau p_k^i x^k + p_j^i q_k^j x^k - q_j^i p_k^j x^k - 2\langle d, x \rangle \gamma^i + 2\langle \gamma, x \rangle d^i = 0. \quad (36)$$

Contracting (36) by  $x^i$  we get

$$\langle d, \gamma \rangle = 2\lambda\tau. \quad (37)$$

Then by (37), (36) is equivalent to

$$2\tau P = PQ - QP - 2R, \quad (R = (r_j^i) \text{ with } r_j^i := \gamma^i d^j - \gamma^j d^i) \quad (38)$$

Plugging  $\mu = 0$  and  $\eta = 0$  into (24) gives

$$-(q_k^i d^k + 4\tau d^i)|x|^2 + 2(4\tau\langle d, x \rangle - \langle d, Qx \rangle)x^i = 0,$$

which is equivalent to

$$Qd = -4\tau d. \quad (39)$$

Since  $Q$  is skew-symmetric, (39) is equivalent to

$$d = 0, \text{ or } \tau = 0 \text{ and } Qd = 0. \quad (40)$$

For (25) and (26), they automatically hold since  $\mu = 0$  and  $\eta = 0$ . Then (22)–(26) are equivalent to (22), (37), (38) and (40), which are broken into one of the following three cases:

$$(\mu = 0, \eta = 0), \quad d = 0, \quad \lambda = 0, \quad Qe = P\gamma, \quad 2\tau P = PQ - QP, \quad (41)$$

$$(\mu = 0, \eta = 0), \quad d = 0, \quad \tau = 0, \quad Qe = -2\lambda\gamma + P\gamma, \quad PQ = QP, \quad (42)$$

$$(\mu = 0, \eta = 0), \quad \tau = 0, \quad Qd = 0, \quad \langle d, \gamma \rangle = 0, \quad Qe = -2\lambda\gamma + P\gamma, \quad PQ - QP = 2R. \quad (43)$$

We see (42)  $\Rightarrow$  (43). So we have only two cases (41) and (43), which are just Theorem 1.2 (i) and (ii) respectively.

**Case B :** Assume  $\eta = -\mu\gamma$  and  $\tau = 0$ . First (26)  $\Leftrightarrow$  (28), which, by  $\eta = -\mu\gamma$ , is written as

$$Qd = -\mu(2\lambda\gamma + P\gamma). \quad (44)$$

Plugging  $\eta = -\mu\gamma$  and  $\tau = 0$  into (23) yields

$$\begin{aligned} 0 &= -2(\langle d, \gamma \rangle + \mu\langle \gamma, e \rangle)x^i + (p_j^i q_k^j - q_j^i p_k^j)x^k \\ &\quad + 2\langle \gamma, x \rangle d^i - 2\langle d, x \rangle \gamma^i + 2\mu(\langle e, x \rangle \gamma^i - \langle \gamma, x \rangle e^i). \end{aligned} \quad (45)$$

Then contracting (45) by  $x^i$  gives

$$\langle d, \gamma \rangle = -\mu\langle \gamma, e \rangle. \quad (46)$$

By (46), (45) is equivalent to

$$PQ - QP = 2R, \quad (R = (r_j^i) \text{ with } r_j^i := \mu(e^i \gamma^j - e^j \gamma^i) + \gamma^i d^j - \gamma^j d^i). \quad (47)$$

Now it follows that (24) automatically holds from  $\eta = -\mu\gamma$ ,  $\tau = 0$ , (22) and (44); (25) automatically holds from  $\eta = -\mu\gamma$ ,  $\tau = 0$ , (46) and (47). So (22)–(26) are equivalent to (22), (44), (46) and (47). This gives Theorem 1.2 (iii). Q.E.D.

We consider two special cases of Theorem 1.2: (ia)  $\rho$  is additionally closed ( $\Leftrightarrow P = 0$  and  $d = \mu e$  in (4)); (ib)  $\rho$  is homothetic ( $\Leftrightarrow d = 0$  and  $\mu = 0$ , or  $d = -\mu e$  and  $\lambda = 0$  in (4)). Then we obtain the following two corollaries respectively.

**Corollary 4.1** *In Theorem 1.2, additionally assume  $\rho$  is a closed 1-form. Then  $V$  is determined by Theorem 1.2 (i) with (6) being replaced by  $Qe = 0$ , or Theorem 1.2 (ii) with (8) being replaced by  $Qe = -2\lambda\gamma$ , or Theorem 1.2 (iii) with (10) being replaced by  $\langle \gamma, e \rangle = 0$  and  $Qe = -2\lambda\gamma$ .*



**Corollary 4.2** *In Theorem 1.2, additionally assume  $\rho$  is homothetic. Then  $V$  is determined by Theorem 1.2 (i), or Theorem 1.2 (ii) with (8) being replaced by*

$$Qe = -2\lambda\gamma + P\gamma, \quad PQ - QP = 0,$$

*or Theorem 1.2 (iii) with (10) being replaced by*

$$Qe = P\gamma, \quad PQ - QP = 2R,$$

*where  $R = (r_j^i)$  is defined by  $r_j^i := 2\mu(e^i\gamma^j - e^j\gamma^i)$ .*

## 5 Isotropic S-curvature

In this section, we use Theorems 1.1 and 1.2 to study conformal vector fields on all  $(\alpha, \beta)$ -spaces of isotropic S-curvature.

The S-curvature is one of the most important non-Riemannian quantities in Finsler geometry which was originally introduced for the volume comparison theorem ([11]). The S-curvature is said to be isotropic if there is a scalar function  $\theta = \theta(x)$  on  $M$  such that

$$\mathbf{S} = (n+1)\theta F.$$

If  $\theta$  is a constant, then we call  $F$  is of constant S-curvature.

In this section, we mainly prove the following theorem.

**Theorem 5.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $n(\geq 2)$ -dimensional non-Riemannian  $(\alpha, \beta)$ -metric on the manifold  $M$ , where  $\phi(0) = 1$ . Suppose  $F$  is of isotropic S-curvature. Then any conformal vector field of  $(M, F)$  is homothetic.*

In [6], Huang-Mo prove Theorem 5.1 when  $F = \alpha + \beta$  is a Randers metric. But our proof here for a Randers metric is quite different from that in [6]. To prove Theorem 5.1, we first show the following lemma.

**Lemma 5.2** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $n(\geq 2)$ -dimensional non-Riemannian  $(\alpha, \beta)$ -metric on the manifold  $M$ , where  $\phi(0) = 1$ . Suppose  $F$  is of isotropic S-curvature. Then we have only three classes:*

- (i) ([3])  $(n \geq 2)$   $F$  is of Randers type in the equivalent form  $F = \alpha + \beta$  satisfying

$$r_{ij} = 2\theta(a_{ij} - b_i b_j) - b_i s_j - b_j s_i, \quad (48)$$

*where  $\theta$  is a scalar function. In this case, the S-curvature is given by  $\mathbf{S} = (n+1)\theta F$ .*

- (ii) ([4])  $(n \geq 2)$   $\phi(s)$  is arbitrary and  $r_{ij} = 0, s_i = 0$ . In this case,  $\mathbf{S} = 0$ .

- (iii) ([5])  $(n = 2)$   $\phi(s)$  is given by

$$\phi(s) = \left\{ (1 + k_1 s^2)(1 + k_2 s^2) \right\}^{\frac{1}{4}} e^{\int_0^s \theta(s) ds}, \quad (49)$$

*and  $\beta$  satisfies*

$$r_{ij} = \frac{3k_1 + k_2 + 4k_1 k_2 b^2}{4 + (k_1 + 3k_2)b^2} (b_i s_j + b_j s_i), \quad (50)$$

*where  $\theta(s)$  is defined by*

$$\theta(s) := \frac{\pm \sqrt{k_2 - k_1}}{2(1 + k_1 s^2)\sqrt{1 + k_2 s^2}}, \quad (51)$$

*and  $k_1$  and  $k_2$  are constants with  $k_2 > k_1$ . In this case,  $\mathbf{S} = 0$ .*

We should note that the condition  $r_{ij} = 0, s_i = 0$  is a special case of (50). In Lemma 5.2, the second class is almost trivial, and if  $F$  is not of Randers type, the third class is essential and in this case, the norm  $\|\beta\|_\alpha$  may not be a constant. In [20], we further determine the local structure of the third class in Lemma 5.2, and it is an Einstein metric but generally not Ricci-flat.

*Proof of Theorem 5.1 :* Define a Riemann metric  $h$  and a 1-form  $\rho$  by (3). We will show by (48) or (50), there are functions  $u, v, w$  such that  $\rho$  is a conformal 1-form of  $h$ .

**Case I:** Suppose that (48) holds. In this case, define in (3)

$$u(b^2) = k_2 w(b^2), \quad v(b^2) = \left(k_1 - \frac{k_1 + k_2}{b^2}\right) w(b^2), \quad (52)$$

where  $k_1, k_2$  are constant and  $w(b^2)$  is some function such that  $h$  is a Riemann metric. For  $u, v, w$  defined by (52), the equation (48) is equivalently transformed into

$$\tilde{r}_{ij} = -\frac{2\theta}{k_1} \left( \frac{1}{1-b^2} + \frac{b^2 w'(b^2)}{w(b^2)} \right) h_{ij}, \quad (53)$$

where we define  $\tilde{r}_{ij} := (p_{i|j} + p_{j|i})/2$  for the covariant derivative of  $\rho$  with respect to  $h$ . So by (53),  $\rho$  is a conformal 1-form with respect to  $h$ . Therefore, by Theorem 1.1, we get the proof of Theorem 5.1 in this case. In particular, if  $k_1, k_2$  and  $w(b^2)$  are taken as  $k_1 = 1, k_2 = -1$  and  $w(b^2) = b^2 - 1$  in (52), then we get the navigation data for a Randers metric.

**Case II:** Suppose that (50) holds. We have shown in [20] that by choosing

$$u(b^2) = 1, \quad v(b^2) = 0, \quad w(b^2) = (1 + k_1 b^2)^{-\frac{3}{4}} (1 + k_2 b^2)^{-\frac{1}{4}},$$

the equation (50) is equivalent to

$$\tilde{r}_{ij} = 0, \quad (54)$$

where we define  $\tilde{r}_{ij} := (p_{i|j} + p_{j|i})/2$  for the covariant derivative of  $\rho$  with respect to  $h$ . The equation (54) shows that  $\rho$  is a Killing form of  $h$  (a special case of conformality). Thus by Theorem 1.1, we get the proof of Theorem 5.1 in this case. Q.E.D.

A Randers metric will be of isotropic S-curvature under some curvature conditions, for example, (ia)  $F$  is a weak Einstein metric ([15]); (ib)  $F$  is Ricci-reversible ([14]); (ic)  $F$  is of Einstein-reversibility ([19]). Then we have the following corollary.

**Corollary 5.3** *On a manifold  $M$ , if a Randers metric  $F = \alpha + \beta$  is weakly Einsteinian, or Ricci-reversible, or of Einstein-reversibility, then any conformal vector field of  $(M, F)$  is homothetic.*

In [15], Shen-Yidirim show that if a Randers metric is of weakly isotropic flag curvature in dimension  $n \geq 3$ , then  $h$  is of constant sectional curvature and  $\rho$  is a conformal 1-form by choosing the navigation data  $u(b^2) = 1 - b^2, v(b^2) = b^2 - 1, w(b^2) = b^2 - 1$  in (3). Thus we have the following corollary.

**Corollary 5.4** *On a manifold  $M$  of dimension  $n \geq 3$ , if a Randers metric  $F = \alpha + \beta$  is of weakly isotropic flag curvature, then any conformal vector field  $V$  of  $(M, F)$  is homothetic, and locally  $V$  can be determined by Theorem 1.2.*

In[2], Bao-Robles-Shen prove that if a Randers metric is of constant flag curvature, then  $h$  is of constant sectional curvature and  $\rho$  is a homothetic 1-form by choosing the navigation data  $u(b^2) = 1 - b^2$ ,  $v(b^2) = b^2 - 1$ ,  $w(b^2) = b^2 - 1$  in (3). Thus we have the following corollary.

**Corollary 5.5** *On a manifold  $M$  of dimension  $n \geq 3$ , if a Randers metric  $F = \alpha + \beta$  is of constant flag curvature, then any conformal vector field  $V$  of  $(M, F)$  is homothetic, and locally  $V$  can be determined by Corollary 4.2.*

## 6 Douglas metrics

In this section, we study conformal vector fields of  $(\alpha, \beta)$ -spaces of Douglas type. We obtain the following theorem.

**Theorem 6.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $n(\geq 2)$ -dimensional  $(\alpha, \beta)$ -metric of non-Randers type on the manifold  $M$ , where  $\phi(0) = 1$ . Suppose  $F$  is Douglas metric. Then any conformal vector field of  $(M, F)$  is homothetic.*

A Randers metric  $F = \alpha + \beta$  is a Douglas metric iff.  $\beta$  is closed ([1]). A conformal vector field on a Randers space of Douglas type is not necessarily homothetic (see Example 7.2 below). To prove Theorem 6.1, we first show the following lemma.

**Lemma 6.2** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $n$ -dimensional  $(\alpha, \beta)$ -metric, where  $\phi(0) = 1$ . Suppose that  $\beta$  is not parallel with respect to  $\alpha$  and  $F$  is not of Randers type. Then we have*

(i) ([8]) ( $n \geq 3$ )  $F$  is a Douglas metric if and only if

$$\begin{aligned} \{1 + (k_1 + k_3)s^2 + k_2s^4\}\phi''(s) &= (k_1 + k_2s^2)\{\phi(s) - s\phi'(s)\}, \\ b_{i|j} &= \theta\{(1 + k_1b^2)a_{ij} + (k_2b^2 + k_3)b_ib_j\}, \end{aligned} \quad (55)$$

where  $k_1, k_2, k_3$  are constant with  $k_2 \neq k_1k_3$ , and  $\theta$  is a scalar function.

(i) ([18]) ( $n = 2$ )  $F$  is a Douglas metric if and only if

$$\begin{aligned} F \text{ is of the metric type } F_0^\pm &= \alpha \pm \beta^2/\alpha, \\ r_{ij} &= \theta\{(1 \pm 2b^2)a_{ij} \mp 3b_ib_j\} + \frac{3}{\pm 1 - b^2}(b_is_j + b_js_i), \end{aligned} \quad (56)$$

where  $\theta$  is a scalar function.

*Proof of Theorem 6.1 :* Define a Riemann metric  $h$  and a 1-form  $\rho$  by (3). We will show by (55) or (56), there are functions  $u, v, w$  such that  $\rho$  is a conformal 1-form of  $h$ .

**Case I:** Suppose that (55) holds. In this case, let  $u, v, w$  in (3) be defined by

$$\begin{aligned} u(b^2) &= 1, \quad v(b^2) = 0, \\ w(b^2) &= e^{-\int_0^{b^2} \frac{1}{2} \frac{k_3 + k_2t}{1 + (k_1 + k_3)t + k_2t^2} dt} \end{aligned}$$

Then it is shown in [18] that (55) is equivalent to

$$p_{i|j} = \theta w(b^2)(1 + k_1b^2)h_{ij}. \quad (57)$$

So by (57),  $\rho$  is a (closed and ) conformal 1-form with respect to  $h$ . Therefore, by Theorem 1.1, we get the proof of Theorem 6.1 in this case.

**Case II:** Suppose that (56) holds. In this case, let  $u, v, w$  in (3) be defined by

$$u(b^2) = \frac{(1 \mp b^2)^3}{(1 \pm 2b^2)^{3/2}}, \quad v(b^2) = \frac{9}{8b^2} \left\{ (1 \pm 2b^2)^{3/2} - \frac{1 \mp 2b^2 + 4b^4}{(1 \pm 2b^2)^{3/2}} \right\}, \quad w(b^2) = 1.$$

Then it is shown in [18] that (56) is equivalent to

$$\tilde{r}_{ij} = \frac{\theta(1 \mp b^2)^2}{(1 \pm 2b^2)^{5/2}} h_{ij}, \quad (58)$$

where we define  $\tilde{r}_{ij} := (p_{i|j} + p_{j|i})/2$  for the covariant derivative of  $\rho$  with respect to  $h$ . So by (58),  $\rho$  is a conformal 1-form with respect to  $h$ . Therefore, by Theorem 1.1, we get the proof of Theorem 6.1 in this case. Q.E.D.

## 7 Projectively flat metrics

For a locally projectively flat  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  of non-Randers type with the dimension  $n \geq 3$ , the local solution of a conformal vector field has been obtained in [21] (also see Corollary 4.1).

In the following, we consider the conformal vector fields on locally projectively flat Randers spaces. In this case, we cannot determine the local structure of those conformal vector fields, and the conformal vector fields on such spaces are not necessarily homothetic.

A Randers metric  $F = \alpha + \beta$  is locally projectively flat if and only if  $\alpha$  is of constant sectional curvature and  $\beta$  is closed ([1]). In this case, we don't need to deform  $\alpha$  and  $\beta$  by (3), or namely, we just choose  $u = w = 1, v = 0$  in (3). To study conformal vector fields on locally projectively flat Randers spaces, we first show a lemma.

**Lemma 7.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $n(\geq 2)$ -dimensional non-Riemannian  $(\alpha, \beta)$ -metric on the manifold  $M$ , where  $\phi(0) = 1$ . Suppose  $\beta$  is closed and  $V$  is a conformal vector field with the conformal factor  $c$ . Then we have  $c_i = \theta b_i$ , where  $c_i := c_{x^i}$  and  $\theta = \theta(x)$  is a scalar function.*

*Proof :* By Lemma 2.3, we have (16). Then since  $\beta$  is closed, the second equation in (16) is written as

$$H_i = -2cb_i, \quad (H := b^i V_i, \quad H_i := H_{x^i}).$$

Therefore, we have

$$0 = H_{i;j} - H_{j;i} = (-2c_j b_i - 2cb_{i;j}) - (-2c_i b_j - 2cb_{j;i}) = -2(b_i c_j - b_j c_i),$$

which imply  $c_i b_j = c_j b_i$ . Thus we have  $c_i = \theta b_i$  for some scalar function  $\theta = \theta(x)$  since  $\beta \neq 0$ . Q.E.D.

Now let  $F = \alpha + \beta$  be locally projectively flat on a manifold  $M$  with the dimension  $n \geq 2$ . Locally express

$$\alpha = \frac{2}{1 + \mu|x|^2} |y|,$$

and put  $V = V^i \partial / \partial x^i$  by (19), namely,

$$V^i = -2(\tau + \langle \eta, x \rangle) x^i + |x|^2 \eta^i + q_r^i x^r + \gamma^i. \quad (59)$$

By Lemma 2.2,  $V$  is a conformal vector field of  $(M, \alpha)$  with the conformal factor  $c$  given by (20), namely,

$$c = \frac{\tau(1 - \mu|x|^2) + \langle \mu\gamma + \eta, x \rangle}{1 + \mu|x|^2} \quad (60)$$

By Lemma 2.3, if the above  $V$  is also a conformal vector field of  $(M, F)$ , then we must have the second equation in (16), namely,

$$V^j \frac{\partial b_i}{\partial x^j} + b_j \frac{\partial V^j}{\partial x^i} = -2cb_i. \quad (61)$$

By Lemma 7.1, we may put

$$b_i = f(c)c_i, \quad (62)$$

where  $f$  is some function. Now plug (59), (60) and (62) into (61), and then we obtain

$$A_1|x|^2 + A_2 = 0, \quad (63)$$

where

$$\begin{aligned} A_1 &= (2\mu(c + \tau)x^i - \eta^i - \mu\gamma^i)(2\mu c^2 - 2\mu\tau c - 4\mu\tau^2 - |\eta|^2 - \mu\langle \eta, \gamma \rangle)f'(c) \\ &\quad + \mu[2(\tau - c)\eta^i - 2\mu(\tau + c)\gamma^i + 4\mu(c^2 - \tau^2)x^i - q_k^i \eta^k - \mu q_k^i \gamma^k]f(c), \\ A_2 &= \left\{ 2\mu[\tau f'(c) + c f'(c) + f(c)]\langle 4\mu\tau\gamma + Q\eta + \mu Q\gamma, x \rangle - 2\mu(\tau + c)(2\tau c - 2c^2 \right. \\ &\quad \left. + \mu|\gamma|^2 + \langle \eta, \gamma \rangle)f'(c) - 2(\mu^2|\gamma|^2 - |\eta|^2 - 2\mu\tau^2 - 2\mu c^2)f(c) \right\} x^i - \\ &\quad (\eta^i + \mu\gamma^i)f'(c)\langle 4\mu\tau\gamma + Q\eta + \mu Q\gamma, x \rangle + (\eta^i + \mu\gamma^i)(2\tau c - 2c^2 + \mu|\gamma|^2 + \langle \eta, \gamma \rangle)f'(c) \\ &\quad + [2(\tau - c)\eta^i - 2\mu(\tau + c)\gamma^i - q_k^i \eta^k - \mu q_k^i \gamma^k]f(c), \end{aligned}$$

To construct non-homothetic vector field, we assume  $A_1 = A_2 = 0$ . Then by the expression of  $A_2$ , we first assume

$$Q\eta = -\mu(4\tau\gamma + Q\gamma). \quad (64)$$

Plug (64) into  $A_2$ , and we obtain

$$\begin{aligned} 0 &= -2[\mu(\tau + c)(2\tau c - 2c^2 + \mu|\gamma|^2 + \langle \eta, \gamma \rangle)f'(c) + (\mu^2|\gamma|^2 - |\eta|^2 - 2\mu\tau^2 - 2\mu c^2)f(c)]x^i \\ &\quad (\eta^i + \mu\gamma^i)[(2\tau c - 2c^2 + \mu|\gamma|^2 + \langle \eta, \gamma \rangle)f'(c) - 2(c - \tau)f(c)]. \end{aligned} \quad (65)$$

By the second term of the right hand side of (65), we may assume

$$f'(c) = \frac{2(c - \tau)}{2\tau c - 2c^2 + \mu|\gamma|^2 + \langle \eta, \gamma \rangle} f(c). \quad (66)$$

Then we can obtain the function  $f(c)$  by solving the ODE (66). Further, plugging (66) into (65) gives

$$|\eta|^2 = \mu(\mu|\gamma|^2 - 4\tau^2). \quad (67)$$

Conversely, if we assume (64), (66) and (67) hold, we can directly verify that  $A_1 = A_2 = 0$ .

Now by the above construction, we obtain the following example.

**Example 7.2** Let  $F = \alpha + \beta$  be a Randers metric on a manifold  $M$ . Locally, define

$$\begin{aligned}\alpha &:= \frac{2}{1 + \mu|x|^2}|y|, & \beta &:= f(c)c_{x^i}, \\ c &:= \frac{\tau(1 - \mu|x|^2) + \langle \mu\gamma + \eta, x \rangle}{1 + \mu|x|^2}, \\ V^i &:= -2(\tau + \langle \eta, x \rangle)x^i + |x|^2\eta^i + q_r^i x^r + \gamma^i,\end{aligned}$$

where  $f$  is a function,  $\mu (\neq 0)$  and  $\tau$  and  $\eta = (\eta^i)$  and  $\gamma = (\gamma^i)$  are of constant values, and  $Q = (q_j^i)$  is a constant skew-symmetric matrix, and those parameters satisfy (64), (66) and (67). It is easy to see that  $F$  is locally projectively flat.

The above construction has shown that the vector field  $V = (V^i)$  is conformal on  $(M, F)$  with the conformal factor  $c$ , and  $V$  is not homothetic.

In Example 7.2, if we take  $Q = 0$  and  $\gamma = 0$ , then we have

$$\begin{aligned}|\eta|^2 &= -4\mu\tau^2, & c &= \frac{\tau(1 - \mu|x|^2) + \langle \eta, x \rangle}{1 + \mu|x|^2}, \\ \beta &= \frac{1}{\tau(1 - \mu|x|^2) + \langle \eta, x \rangle} \left\{ \langle \eta, y \rangle - \frac{2\mu(2\tau + \langle \eta, x \rangle)\langle x, y \rangle}{1 + \mu|x|^2} \right\}, \\ V^i &= -2(\tau + \langle \eta, x \rangle)x^i + |x|^2\eta^i,\end{aligned}$$

where we have put  $f(c) = 1/c$  (by (66)). Thus we obtain a simple family of non-homothetic conformal vector fields on a corresponding family of locally projectively flat Randers spaces, which have been shown in [21].

In a more general case, Example 7.2 also shows the following remark.

**Remark 7.3** In Theorem 1.1, if we assume  $h$  is of constant non-zero sectional curvature, and  $\rho$  is just closed (instead of being conformal), then the conformal vector field of  $(M, F)$  is not necessary homothetic.

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